Mathematics for Purchasing

Matrices and Matrix Calculations

MSc Supply Chain & Purchasing Management

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Résumé

This document is an introduction to matrix calculus and matrix manipulation, which will be essential when you're called upon to manipulate data tables in future courses and in your future career.

You are not expected to master the subject or the abstract concept, but simply to understand how calculations can be performed with such objects. We will be working with them in the context of linear regression to show you just how powerful these tools can be. From a computing and computational point of view, they offer many advantages, making calculations much simpler and faster.

You will find a few exercises at the end of this document that you can do to practice and see if you've understood how these objects work.

In the following, $\mathbb{R}$ denotes the set of real numbers and $\mathbb{R}^n$ designates an n-tuple of real numbers, *i.e.*, a set of n real numbers.

1 Definitions and Special Matrices

There is no need to be afraid of these objects, as you have already handled them when working with Excel.

Définition 1.1: Matrice

A matrix $\mathbf{A} \in \mathscr{M}_{n,p}(\mathbb{R})$ is a *table with n lines and p columns* where each cell contains a real number. Such a matrix, denoted by $\mathbf{A}$, can be seen as

$$\mathbf{A} = (a_{ij})_{i,j=1}^{n,p} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ a_{21} & a_{22} & \cdots & a_{2p} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{np} \end{pmatrix}.$$

In this definition, note that the first subscript, subscript i, refers to the i-th row of the table. The second subscript, subscript j, refers to the j-th column of our table.

The matrix should therefore be seen as a table of values. This vision is very important for what follows, as it corresponds to the representation most commonly used when representing data with the following conventions :

- **on line :** we represent the individuals (total n, each described by p attributes).
- **in column :** we represent the values taken by the different attributes (total p) on the different elements of the sample (which are our individuals and number n).

Another special case are matrices of type $(1, n)$ and $(n, 1)$ which designate **vectors** or **matrices** rows and columns respectively.

Définition 1.2: Row and Column Vectors

Let $\mathbf{A}$ be a matrix of type (n, p). For any $i \in [\![1, n]\!]$, we call *i-th row vector of* $\mathbf{A}$ the vector of $\mathbb{R}^p$ defined by

$$L_i = (a_{i1}, a_{i2}, \ldots, a_{ip}).$$

For any $j \in [\![1, p]\!]$, we call *j-th column vector of* $\mathbf{A}$ the vector of $\mathbb{R}^n$ defined by

$$C_j = (a_{1j}, a_{2j}, \ldots, a_{pj}).$$

Définition 1.3: Square Matrices

For any $n > 0$, a matrix that has the same number of rows and columns is called a **square matrix**. This type of matrix is said to be of order n. A square matrix $\mathbf{A}$ is said to be **diagonal** if for any $i \neq j$, $a_{ij} = 0$.
A square matrix $\mathbf{A}$ is said to be **inferior triangular** (respectively **superior triangular**) if for all $i > j$ (respectively for all $i < j$) $a_{ij} = 0$.

Example 1.1. *The following matrices* $\mathbf{A}, \mathbf{B}, \mathbf{C}$ *and* $\mathbf{D}$ *are any* $(3, 3)$ *square matrices, diagonal, upper triangular and lower triangular respectively.*

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & \pi \end{pmatrix}, \quad \mathbf{C} = \begin{pmatrix} e & 3 & \ln(2) \\ 0 & 1 & 0 \\ 0 & 0 & \pi \end{pmatrix}, \quad \mathbf{D} = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & -2 & 0.12 \end{pmatrix}.$$

The literature is full of square matrices of size n with very interesting properties.

A final important matrix, but one that is very simple to manipulate, is the identity matrix. It's a matrix that plays the same role as the 1 value in classical multiplication, but we will come back to that later, and it can be written very simply, as the following definition shows.

Définition 1.4: Identity Matrix

We call **identity matrix of order** n, denoted I_n, the diagonal square matrix of order n whose diagonal coefficients are all equal to 1 and the other coefficients are equal to 0. So we have :

$$\mathbf{I}_n = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 1 \end{pmatrix}.$$

Just as we do with the real numbers we manipulate. You can add two matrices together and multiply a matrix by a number.

The addition between two matrices $A, B \in \mathscr{M}_{n,p}(\mathbb{K})$ and the multiplication of a matrix by a real number *ambda* are defined by

$$\mathbf{A} + \mathbf{B} = \begin{pmatrix} a_{11} & \dots & a_{1p} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{np} \end{pmatrix} + \begin{pmatrix} b_{11} & \dots & b_{1p} \\ \vdots & & \vdots \\ b_{n1} & \dots & b_{np} \end{pmatrix} = \begin{pmatrix} a_{11} + b_{11} & \dots & a_{1p} + b_{1p} \\ \vdots & & \vdots \\ a_{n1} + a_{n1} & \dots & a_{np} + b_{np} \end{pmatrix},$$

$$\lambda \mathbf{A} = \lambda \begin{pmatrix} a_{11} & \dots & a_{1p} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{np} \end{pmatrix} = \begin{pmatrix} \lambda a_{11} & \dots & \lambda a_{1p} \\ \vdots & & \vdots \\ \lambda a_{n1} & \dots & \lambda a_{np} \end{pmatrix}.$$

Example 1.2. *We consider the matrices* **A** *and* **B** *defined as follow :*

$$\mathbf{A} = \begin{pmatrix} 2 & 4 \\ 1 & -2 \\ -5 & 3 \end{pmatrix} \quad \textit{and} \quad \mathbf{B} = \begin{pmatrix} -1 & 1 \\ 0 & 3 \\ -6 & 4 \end{pmatrix}.$$

Then, we have :

$$\mathbf{A} + \mathbf{B} = \begin{pmatrix} 2 & 4 \\ 1 & -2 \\ -5 & 3 \end{pmatrix} + \begin{pmatrix} -1 & 1 \\ 0 & 3 \\ -6 & 4 \end{pmatrix} = \begin{pmatrix} 2-1 & 4+1 \\ 1+0 & -2+3 \\ -5-6 & 3+4 \end{pmatrix} = \begin{pmatrix} 1 & 5 \\ 1 & 1 \\ -11 & 7 \end{pmatrix}$$

and

$$3\mathbf{A} = 3 \begin{pmatrix} 2 & 4 \\ 1 & -2 \\ -5 & 3 \end{pmatrix} = \begin{pmatrix} 3 \times 2 & 3 \times 4 \\ 3 \times 1 & 3 \times (-2) \\ 3 \times (-5) & 3 \times 3 \end{pmatrix} = \begin{pmatrix} 6 & 1 \\ 3 & -6 \\ -15 & 9 \end{pmatrix}$$

Définition 1.5: Matrix Transposition

Let $\mathbf{A} \in \mathscr{M}_{n,p}(\mathbb{K})$, we call **transposed of matrix A**, denoted $\mathbf{A}^\top$, matrix $\mathbf{A}'$ of $\mathscr{M}_{p,n}(\mathbb{R})$ defined for all $(i,j) \in [\![1,p]\!] \times [\![1,n]\!]$ by $a'_{ij} = a_{ji}$. Otherwise written

$$\text{if } \mathbf{A} = \begin{pmatrix} a_{11} & \dots & a_{1p} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{np} \end{pmatrix} \quad \text{then} \quad \mathbf{A}^T = \begin{pmatrix} a_{11} & \dots & a_{n1} \\ \vdots & & \vdots \\ a_{1p} & \dots & a_{np} \end{pmatrix}$$

Note that if the matrix $\mathbf{A}$ has n lines and p columns, its transpose has p lines and n columns. Transposition is a transformation that admits the following properties

Example 1.3. *Let* $\mathbf{A}$ *be a matrix with* 2 *lines and* 4 *columns defined by*

$$\mathbf{A} = \begin{pmatrix} 4 & 5 & 2 & 1 \\ 5 & -1 & -3 & 2 \end{pmatrix}.$$

Then its transpose $\mathbf{A}'$ *is a matrix with* 4 *lines and* 2 *columns defined by*

$$\mathbf{A}' = \mathbf{A}^\top = \begin{pmatrix} 4 & 5 \\ 5 & -1 \\ 2 & -3 \\ 1 & 2 \end{pmatrix}.$$

Proposition 1.1: Properties of Transposition

Let $\mathbf{A}, \mathbf{B}$ be two $\mathscr{M}_{m,n}(\mathbb{R})$ matrices, $\mathbf{C}$ be a $\mathscr{M}_{n,p}(\mathbb{R})$ matrix, and let $\lambda \in \mathbb{K}$ be a scalar, then :

i) $(\lambda\mathbf{A})^\top = \lambda\mathbf{A}^\top$,
ii) $(\mathbf{A}+\mathbf{B})^\top = \mathbf{A}^\top + \mathbf{B}^\top$,
iii) $(\mathbf{AC})^\top = \mathbf{C}^\top\mathbf{A}^\top$.

The last property shows that the transpose of the product of two matrices is equal to the product of the transposes of the two matrices in which the order of the terms has been exchanged.

The case of square matrices is more interesting to study and highlights an important property of certain matrices : **symmetry**.

Définition 1.6: Symmetrical and Anti-Symmetrical Matrices

Let $\mathbf{A} \in \mathscr{M}_n(\mathbb{R})$, then :

i) $\mathbf{A}$ is said to be symmetric if $\mathbf{A}^T = \mathbf{A}$,

ii) $\mathbf{A}$ is said to be anti-symmetrical if $\mathbf{A}^T = -\mathbf{A}$

Example 1.4. *The following matrices* $\mathbf{S}$ *and* $\mathbf{A}$ *are symmetrical and anti-symmetrical matrices of order* 4 *respectively.*

$$\mathbf{S} = \begin{pmatrix} a & b & c & d \\ b & e & f & g \\ c & f & h & i \\ d & g & i & j \end{pmatrix} \quad et \quad \mathbf{A} = \begin{pmatrix} 0 & -b & -c & -d \\ b & 0 & -f & -g \\ c & f & 0 & -i \\ d & g & i & 0 \end{pmatrix}.$$

Note that being anti-symmetrical necessarily imposes that the diagonal of the matrix be zero.

2 Operations on matrices

We present some operations on matrices that can be useful for computations on matrices.

Définition 2.1: Basic Operations

We name **basic operations on rows (or on columns)** on a matrix one of the following operation :

i) multiply a row (or a column) by a scalar λ, *i.e.* do the operation $L_i \leftarrow \lambda L_i$ (or $C_i \leftarrow \lambda C_i$),

ii) add to a given row (or a column) the product of another row (or from another column) multiplied by a scalar λ, *i.e.* $L_i \leftarrow L_i + \lambda L_j$ (ou $C_i \leftarrow C_i + \lambda C_j$),

iii) exchange two rows (or two columns), *i.e.* $L_i \leftrightarrow L_j$ (ou $C_i \leftrightarrow C_j$).

These transformations can be rewritten using matrices. For this, we consider a matrix $\mathbf{A} \in \mathscr{M}_{n,p}(\mathbb{K})$ on which some operations are done on its rows.

i) **Operation** $L_i \leftarrow \lambda L_i$, is the same as doing the following product :

$$
\begin{array}{c}
\\
i \rightarrow
\end{array}
\begin{pmatrix}
1 & & & & & \\
& \ddots & & & 0 & \\
& & 1 & & & \\
& & & \lambda & & \\
& 0 & & & \ddots & \\
& & & & & 1
\end{pmatrix}
\mathbf{A}
\quad \text{(column } i \downarrow \text{ at } \lambda\text{)}
$$

ii) **Operation** $L_i \leftarrow L_i + \lambda L_j$, is the same as doing the following product :

$$
i \rightarrow
\begin{pmatrix}
1 & & & & & & & & \\
& \ddots & & & & & 0 & & \\
& & 1 & & & & & & \\
0 & \cdots & 0 & 1 & 0 & \ldots & \lambda & \cdots & 0 \\
& & & & 1 & & & & \\
& & & & & \ddots & & & \\
& 0 & & & & & \ddots & & \\
& & & & & & & 1 & \\
& & & & & & & & 1
\end{pmatrix}
\mathbf{A}
\quad \text{(column } j \downarrow \text{ at } \lambda\text{)}
$$

iii) **Operation** $L_i \leftrightarrow L_j$, is the same as doing the following product :

$$
\begin{pmatrix}
1 & & & & & & & \\
& \ddots & & & & & & \\
& & 0 & \cdots & \cdots & \cdots & 1 & \\
& & \vdots & 1 & & & \vdots & \\
& & \vdots & & \ddots & & \vdots & \\
& & \vdots & & & 1 & \vdots & \\
& & 1 & \cdots & \cdots & \cdots & 0 & \\
& & & & & & & \ddots & \\
& & & & & & & & 1
\end{pmatrix}
\mathbf{A}
\quad \text{(rows and columns } i \text{ and } j \text{ marked by } i \rightarrow,\ j \rightarrow,\ i \downarrow,\ j \downarrow\text{)}
$$

Remark. We can see that performing an elementary operation on the rows means pre-multiplying the matrix $\mathbf{A}$ by the elementary operation. If you want to work on the columns, you have to post-multiply the matrix $\mathbf{A}$ by the elementary operation matrix.

Beware, however, of the size of the matrices representing the elementary operations, depending on whether you are working on rows or columns!

3 Rank, determinant and matrix inverse

3.1 Rank of a matrix

Just as we defined the rank of a linear application, we can also define the rank of a matrix.

Définition 3.1: Rank of a matrix

If $\mathbf{A}$ is a $\mathscr{M}_{n,p}(\mathbb{K})$ matrix, we call $rg(\mathbf{A})$ the rank of the family of column vectors of $\mathbf{A}$ that are $\mathbb{K}^n$ vectors.

We are going to see that the rank of a matrix $\mathbf{A}$ is the number of independent columns of a given matrix.

Why are we interested in column vectors? Because column vectors represent the images of the base vectors of the vector space E, so the rank of the family of column vectors of the matrix $\mathbf{A}$ will determine the dimension of the image space of the linear application under study.

Proposition 3.1: Rank and base

Let E be a finite-dimensional $\mathbb{K}$-vector space n provided with a basis $\mathscr{B} = (\mathbf{e}_1, \ldots, \mathbf{e}_n)$ and let $(\mathbf{x}_1, \ldots, \mathbf{x}_p)$ be a family of vectors of E, then :

$$rg(\mathbf{x}_1, \ldots, \mathbf{x}_n) = rg(\underset{\mathscr{B}}{Mat}\,(\mathbf{x}_1, \ldots, \mathbf{x}_n)).$$

From this result we immediately deduce that any square matrix $\mathbf{A}$ of order n is invertible if only its rank is equal to n.

We end this introduction to the rank of a matrix with the following result

Proposition 3.2: Properties of the rank

Let $\mathbf{A} \in \mathscr{M}_{n,p}(\mathbb{K})$, then $rg(\mathbf{A}^\top) = rg(\mathbf{A})$. The rank of a matrix is invariant by transposition.

3.2 Matrix Determinant

A proper presentation of the determinant of a matrix would require the introduction of various more or less abstract concepts, which will not be useful for the rest of this course. So we won't even present a proper definition of the determinant, and will confine ourselves to giving a few properties concerning it.

Readers wishing to know more about the construction of the determinant are invited to study the *symmetric group* and the *alternating n-linear forms*. We will only give the definition of the determinant for a matrix, with a minimum of explanation.

Définition 3.2: Determinant

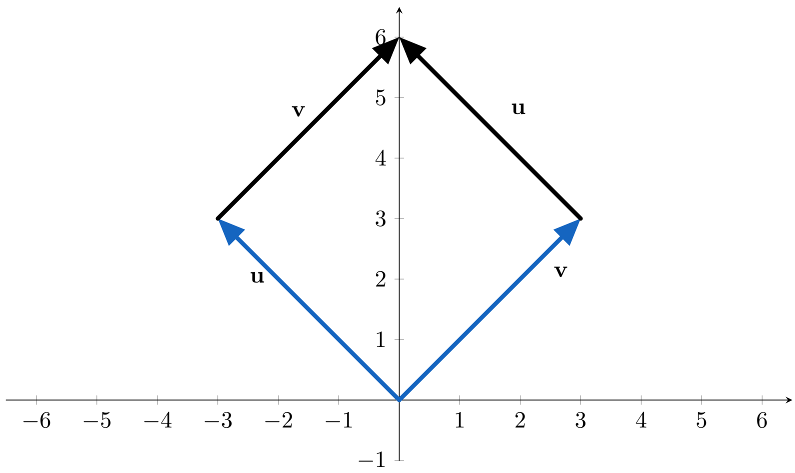
Let $\mathbf{A}$ be a $\mathscr{M}_n(\mathbb{R})$ matrix whose elements are denoted $(a_{ij})_{i,j=1}^n$. Let $\mathfrak{S}_n$ be the group of permutations of $[\![1,n]\!]$, i.e. the set of bijections of $[\![1,n]\!]$ into $[\![1,n]\!]$ (like a transposition that swaps places between two elements i and j). We define the **determinant** of the matrix $\mathbf{A}$ by the relation

$$det(\mathbf{A}) = \sum_{\sigma \in \mathfrak{S}_n} \varepsilon(\sigma) \prod_{i=1}^{n} a_{\sigma(i),i},$$

where $\varepsilon(\sigma)$ is called **the permutation signature** which is equal to ± 1.

The determinant is a value that can be associated with a family of vectors, finite-dimensional linear applications or, more generally, matrices.
In an algebraic context, it is used to determine whether a family of vectors constitutes a basis of a vector space.

Remark This number can also be interpreted geometrically. In spaces of dimension 2 or 3, it represents the area of a parallelogram or the volume of a parallelepiped in space generated by families of two or three vectors respectively.



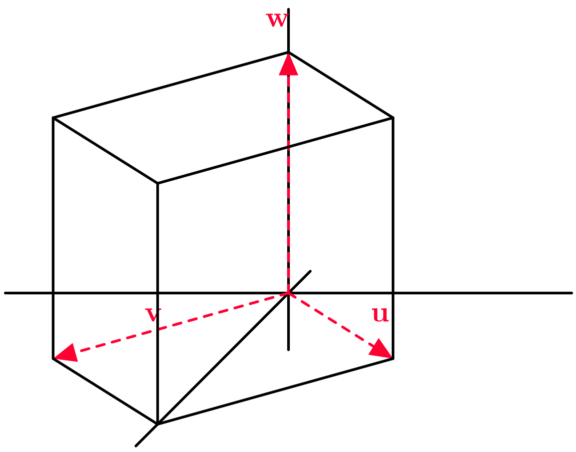
In this first example the vectors $(\mathbf{u}, \mathbf{v})$ are defined by $\mathbf{u} = (-3, 3)$ and $\mathbf{v} = (3, 3)$ which generates a square with an area of 18. We would then have :

$$det(\mathbf{u}, \mathbf{v}) = \begin{vmatrix} 3 & -3 \\ 3 & 3 \end{vmatrix} = 18.$$

Now consider the vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ which are defined by

$$\mathbf{u} = \begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} -3 \\ 3 \\ 0 \end{pmatrix} \quad \text{et} \quad \mathbf{w} = \begin{pmatrix} 0 \\ 0 \\ \sqrt{18} \end{pmatrix}.$$

This generates the following parallelepiped in a 3-dimensional space, whose volume, which is nothing other than the determinant of the families of the 3 vectors, is equal to $18\sqrt{18} = 54\sqrt{2}$.



Indeed :

$$det(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \begin{vmatrix} 3 & -3 & 0 \\ 3 & 3 & 0 \\ 0 & 0 & \sqrt{18} \end{vmatrix} = 18\sqrt{18}.$$

We will see later how to calculate the determinants of such vector families. But first, let us look at some properties of the determinant.

Proposition 3.3: Base and determinant

Let $(\mathbf{x}_1, \ldots, \mathbf{x}_n)$ be a family of vectors of a finite-dimensional n vector space E. Then this family constitutes a basis of this vector space if and only if the determinant associated with this family of vectors is non-zero, $det(\mathbf{x}_1, \ldots, \mathbf{x}_n) \neq \mathbf{0}$..

In the same way, we saw that we could represent u-endomorphisms on a vector space E by determining the images of the vectors of the basis of E by u, the images forming the column vectors of the matrix. We can therefore see if the images form a basis of the space E and thus see if our application is one to one and onto by studying the determinant of this linear application or its associated matrix.

Proposition 3.4: Automorphism characterization

Let $u \in \mathscr{L}(E)$, then u is an automorphism, *i.e.* a bijective endomorphism if and only if $det(u) \neq 0$.

Proposition 3.5: Inverse of a matrix

Let $\mathbf{A} \in \mathscr{M}_n(\mathbb{K})$ then $\mathbf{A}$ is invertible if and only if $det(\mathbf{A}) \neq 0$.

Remark. The determinant only makes sense for square matrices *i.e.* for endomorphisms of vector spaces.

We end this presentation of the determinant with a few important properties.

Proposition 3.6: Determinant properties

Let $\mathbf{A}$ et $\mathbf{B}$ two matrices that belong in $\mathscr{M}_n(\mathbb{K})$, then :

$$det(\mathbf{A}^\top) = det(\mathbf{A}) \quad \text{et} \quad det(\mathbf{AB}) = det(\mathbf{A})det(\mathbf{B}).$$

Furthermore, if $\mathbf{A}$ is invertible, *i.e.* if its determinant is not equal to 0, then :

$$det(\mathbf{A}^{-1}) = \frac{1}{det(\mathbf{A})}.$$

On the other hand, there are no special properties concerning the determinant of the sum of two matrices ! In general

$$det(A + B) \neq det(A) + det(B).$$

These properties will be illustrated on simple examples in the next section on matrix calculations.

3.2.1 Link with elementary operations.

The determinant still has a few properties that will be useful when calculating it :

- it does not change if you add to a column a linear combination of the other columns
- it changes sign if the positions of two columns are exchanged
- if a column is multiplied by a scalar λ, then the determinant is itself multiplied by this scalar λ (because the determinant is linear with respect to each column).

All the comments made about rows also apply to columns, since $det(\mathbf{A}^T) = det(\mathbf{A})$.

3.2.2 Minors of a matrix.

The results in this section will be very useful for the practical calculation of the determinant, and will also help us to explain the inverse of a matrix.

Définition 3.3: Minor and Cofactor

Let $\mathbf{A} = (a_{ij})_{i,j=1}^n \in \mathscr{M}_n(\mathbb{K})$ then, for all couples (i, j)

- we call **minor of A relative to the i-th row and j-th column**, denoted Δ_{ij}, the determinant of the matrix $\mathbf{A}$ without the i-th row and j-th column,
- we call **cofactor of A**, denoted A_{ij}, the scalar defined by $A_{ij} = (-1)^{i+j}\Delta_{ij}$.

Proposition 3.7: Determinant and minors

Let $\mathbf{A} = (a_{ij})_{i,j=1}^n \in \mathscr{M}_n(\mathbb{K})$ then for all $1 \leq j \leq n$, $det(A) = \sum_{i=1}^n (-1)^{i+j} a_{ij}\Delta_{ij} = \sum_{i=1}^n a_{ij}A_{ij}$.

3.2.3 Matrix inverse

We still need to define the comatrix in order to provide an expression for the inverse of a matrix $\mathbf{A} \in \mathscr{M}_n(\mathbb{K})$.

Définition 3.4: Comatrix

Let $\mathbf{A} = (a_{ij})_{i,j=1}^n \in \mathscr{M}_n(\mathbb{K})$, we call **comatrix of A**, notée $Com(\mathbf{A})$, the cofactor matrix of A, *i.e.* $Com(\mathbf{A}) = (a_{ij})_{i,j=1}^n$.

A cofactor of A_{ij} is defined as the determinant of $\mathbf{A}$ for which the i-th row and j-th column are removed.

Proposition 3.8: Matrix inverse

Let $\mathbf{A} \in \mathscr{M}_n(\mathbb{K})$ and let us assume that $\mathbf{A}$ is invertible, then $Com(\mathbf{A})^\top \mathbf{A} = \mathbf{A}^\top Com(\mathbf{A}) = det(\mathbf{A})\mathbf{I}_n$.

We can also rewrite the result of this proposition as follows : if $\mathbf{A}$ is invertible then

$$\mathbf{A}^{-1} = \frac{1}{det(\mathbf{A})} Com(\mathbf{A})^\top.$$

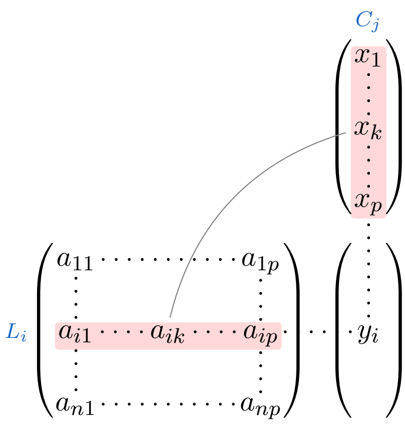
4 Explicit calculations

It is time to put some concrete flesh on the bones of what we have been developing throughout this section. We are going to see how to perform simple operations on matrices and how to multiply matrices.

We start with the product between a matrix $\mathbf{A} = (a_{ij})_{i,j=1}^{n,p} \in \mathscr{M}_{n,p}(\mathbb{R})$ and a vector of $\mathbf{x} = (x_1, x_2, \cdots, x_p)^\top \in \mathbb{R}^p$.

$$\mathbf{Ax} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ a_{21} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ a_{n1} & \cdots & \cdots & a_{np} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_4 \end{pmatrix} = \begin{pmatrix} \sum_{k=1}^p a_{1k} x_k \\ \sum_{k=1}^p a_{2k} x_k \\ \vdots \\ \vdots \sum_{k=1}^p a_{nk} x_k \end{pmatrix}$$

The product of the $\mathbf{A}$ matrix and the $\mathbf{x}$ vector can also be represented more graphically.

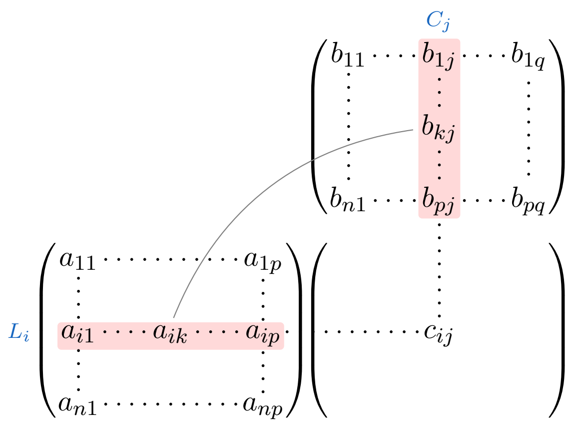


While we saw earlier how to sum two matrices of the same size by summing term by term, multiplication is quite different.
We won't be doing the term product this time (at least, that is not the classic matrix product). Let us take a look and consider $\mathbf{A} = (a_{ij})_{i,j=1}^{n,p} \in \mathscr{M}_{n,p}(\mathbb{R})$ and $\mathbf{B} = (b_{jk})_{j,k=1}^{p,q} \in \mathscr{M}_{p,q}(\mathbb{R})$.

If we denote $\mathbf{C}$ the matrix product $\mathbf{AB}$, then the index element (i, j) of this matrix $\mathbf{C}$ is equal to

$$c_{ij} = (AB)_{ij} = \sum_{k=1}^{p} a_{ik} b_{kj}.$$

We represent this product between the two matrices $\mathbf{A}$ and $\mathbf{B}$ to calculate the coefficient c_{ij}.



Example 4.1. *Let* $\mathbf{A} \in \mathscr{M}_{3,4}(\mathbb{R})$, $\mathbf{B} \in \mathscr{M}_{4,2}(\mathbb{R})$ *et* $\mathbf{x} \in \mathbb{R}^4$ *defined by :*

$$\mathbf{A} = \begin{pmatrix} 2 & -1 & 0 & 4 \\ -1 & 0 & 3 & -2 \\ 5 & 0 & 0 & -4 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 0 & 1 \\ 4 & 3 \\ 0 & -2 \\ 1 & -1 \end{pmatrix} \quad \text{and} \quad \mathbf{x} = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 2 \end{pmatrix}.$$

Then the product of the matrix $\mathbf{A}$ *with the vector* $\mathbf{x}$ *(which amounts to calculating the image of the vector* $\mathbf{x}$ *by the linear application associated with the matrix* $\mathbf{A}$*) is*

$$\mathbf{Ax} = \begin{pmatrix} 2 & -1 & 0 & 4 \\ -1 & 0 & 3 & -2 \\ 5 & 0 & 0 & -4 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \times 1 - 1 \times (-1) + 0 \times 0 + 4 \times 2 \\ \dots \\ \dots \end{pmatrix} = \begin{pmatrix} 10 \\ -5 \\ -3 \end{pmatrix}$$

Then the product $\mathbf{C} = \mathbf{AB}$ *is*

$$\mathbf{C} = \mathbf{AB} = \begin{pmatrix} 2 & -1 & 0 & 4 \\ -1 & 0 & 3 & -2 \\ 5 & 0 & 0 & -4 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 4 & 3 \\ 0 & -2 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 2 \times 0 - 1 \times 4 + 0 \times 0 + 4 \times 1 & \dots \\ \dots & \dots \\ \dots & \dots \end{pmatrix} = \begin{pmatrix} 0 & -5 \\ -2 & -5 \\ -4 & 9 \end{pmatrix}.$$

When performing a matrix product, make sure that the product is **licit** ! A product $\mathbf{AB}$ is legal if the matrix A has as many columns as the matrix $\mathbf{B}$ has rows.

Example 4.2.

Finally, note that the product of two matrices **A** and **B** gives a matrix whose row number equals the row number of matrix **A** and whose column number equals the column number of matrix **B**. Saying differently, the product of a $\mathscr{M}_{n,p}(\mathbb{R})$ matrix with a $\mathscr{M}p, q(\mathbb{R})$ matrix gives a $\mathscr{M}n, q(\mathbb{R})$ matrix.

4.1 Staggered matrix (reduced)

Définition 4.1: Staggered

A matrix is said to be row-scaled if its number of zeros preceding the first non-value of a row increases row by row until it eventually has rows containing only zeros.

Définition 4.2: Reduced Scaled Matrix

A scaled matrix is said to be reduced if it is scaled and if the first non-zero values in each row are equal to 1. These values are called **pivots**.

Remark : we can also define a scaled (reduced) matrix by reasoning about the columns.

Staggered matrices, and especially reduced staggered matrices, will play an important role in the Gauss pivot method, also known as Gauss-Jordan elimination, which we'll look at in the next section.
It is also with the operations performed to obtain such matrices that we can directly determine the rank or inverse of a given matrix (if it is invertible!).

Example 4.3. *The matrices* $\mathbf{A}, \mathbf{B}$ *and* $\mathbf{C}$ *are respectively scaled, reduced scaled and unscaled.*

$$A = \begin{pmatrix} 0 & 4 & 3 & 2 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 8 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 4 & -3 & 5 \\ 0 & 1 & -2 & 7 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad et \quad C = \begin{pmatrix} 2 & 4 & -3 & 5 \\ 4 & 1 & -2 & 7 \\ 6 & 2 & 4 & 1 \\ 8 & -5 & -2 & 1 \end{pmatrix}.$$

4.2 Rank computation

Determining the rank of a matrix couldn't be simpler. Simply perform elementary operations on the rows of a matrix (or on the columns) to obtain a scaled or reduced scaled matrix. The rank of the matrix is then directly equal to the number of non-zero rows in the matrix.

Example 4.4. *Let us take the matrices from the previous example*

$$\mathbf{A} = \begin{pmatrix} 0 & 4 & 3 & 2 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 8 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 1 & 4 & -3 & 5 \\ 0 & 1 & -2 & 7 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \textit{et} \quad \mathbf{C} = \begin{pmatrix} 2 & 4 & -3 & 5 \\ 4 & 1 & -2 & 7 \\ 6 & 2 & 4 & 1 \\ 8 & -5 & -2 & 1 \end{pmatrix}.$$

We can see straight away that the **A** *and* **B** *matrices have ranks of* 3 *and* 4 *respectively. However, the rank of matrix* **C** *is less obvious, so we'll perform some elementary operations on the rows to determine its rank.*

$$\mathbf{C} = \begin{pmatrix} 2 & 4 & -3 & 5 \\ 4 & 1 & -2 & 7 \\ 6 & 2 & 4 & 1 \\ 8 & -5 & -2 & 1 \end{pmatrix}$$

$\downarrow$ *We will use the* 2 *value in position* (1, 1) *to cancel the other lines.*

$$\begin{pmatrix} 2 & 4 & -3 & 5 \\ 0 & -7 & 4 & -3 \\ 0 & -10 & 13 & -18 \\ 0 & -21 & 10 & -19 \end{pmatrix} \begin{matrix} \\ L_2 \leftarrow L_2 - 2L_1 \\ L_3 \leftarrow L_3 - 3L_1 \\ L_4 \leftarrow L_4 - 4L_1 \end{matrix}$$

$\downarrow$ *we then use the* -7 *in position* (2, 2) *to make* 0 *appear in the last two lines.*

$$\begin{pmatrix} 2 & 4 & -3 & 5 \\ 0 & -7 & 4 & -3 \\ 0 & 0 & 51 & -96 \\ 0 & 0 & -2 & -10 \end{pmatrix} \begin{matrix} \\ \\ L_3 \leftarrow 7L_3 - 10L_2 \\ L_4 \leftarrow L_4 - 3L_2 \end{matrix}$$

Note that the last two rows are independent, so the matrix C *has rank* 4*. This example shows the benefits of using reduced scaled matrices, which greatly simplify calculations.*

Remark : The rank of a non-square matrix can also be calculated, as it is simply the dimension of the image space of the associated linear application.
For example, try calculating the rank of the following matrix :

$$\mathbf{A} = \begin{pmatrix} 1 & 4 & 7 & -2 & 0 \\ 3 & -1 & 2 & -9 & 6 \\ 7 & 2 & 4 & 5 & 2 \end{pmatrix}.$$

Remember that the rank of a matrix is always less than or equal to the minimum between the number of columns and the number of rows in the matrix. In this example, the rank cannot exceed 3. In general, if $\mathbf{A}$ is a matrix with n rows and p columns, then :

$$rg(\mathbf{A}) \leq \inf(n, p).$$

4.3 Determinant calculation

The calculation of the determinant is generally very complex, except for very special matrices. On the other hand, it is very simple in low-dimensional spaces, such as 2 and 3, where formulas allow us to calculate the determinant very easily.

- **In 2 dimension :** the determinant of $\mathbf{A} = (a_{ij})_{i,j=1}^2$ is given by :

$$det(\mathbf{A}) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}.$$

- **In 3 dimension :** the determinant of $\mathbf{A} = (a_{ij})_{i,j=1}^2$ is given by :

$$det(\mathbf{A}) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{13}a_{31} + a_{21}a_{32}a_{13}$$
$$- a_{11}a_{13}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}.$$

  This rule is called *Sarrus rule.*

From these two examples, we can see that the determinant is equal to *the sum of products of elements lying on the same extended diagonal* ***minus*** *the sum of products of anti-diagonal elements.* These two relations will be used to calculate the determinant of matrices of larger size, so it is important to memorize them well.

Some particular cases. The determinant is extremely simple to calculate for some matrices, whatever their size. In these special cases, we will try to transform our matrix so that we can easily calculate its determinant.

- **Diagonal matrix :** let $\mathbf{A} \in \mathscr{M}_n(\mathbb{K})$ be a diagonal matrix, *i.e.* $a_{ij} = 0$ for all $i \neq j$, then $det(mA) = \prod_{i=1}^n a_{ii}$.
  It can also be seen as :

$$det(\mathbf{A}) = \begin{vmatrix} a_{11} & 0 & \cdots & \cdots & 0 \\ 0 & a_{22} & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & a_{nn} \end{vmatrix} = a_{11}a_{22}\ldots a_{nn} = \prod_{i=1}^{n} a_{ii}.$$

From this result, we can immediately see that a necessary and sufficient condition for a diagonal matrix to be invertible is that its diagonal elements must be non-zero.

- **Triangular matrix** let $\mathbf{A} \in \mathscr{M}_n(\mathbb{K})$ be an upper triangular matrix (the result remains the same in the case of upper triangular matrices) , *i.e.* $a_{ij} = 0$ for all $i > j$, then $det(\mathbf{A}) = \prod_{i=1}^{n} a_{ii}.\}$.
  It can also be seen as :

$$det(\mathbf{A}) = \begin{vmatrix} a_{11} & a_{12} & \cdots & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & a_{(n-1)1} \\ 0 & 0 & \cdots & 0 & a_{nn} \end{vmatrix} = a_{11}a_{22}\dots a_{nn} = \prod_{i=1}^{n} a_{ii}.$$

  Again, a triangular matrix is invertible if and only if its diagonal elements are non-zero.

Example 4.5. *In the case of* 2 *and* 3 *matrices, we can repeat the examples used to illustrate geometrically what the determinant is.*

Now consider two matrices $\mathbf{A}$ *and* $\mathbf{B}$ *of type* 4*, diagonal and upper triangular respectively :*

$$det(\mathbf{A}) = \begin{vmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 4 \end{vmatrix} = 24 \quad et \quad \mathbf{B} = \begin{vmatrix} 2 & 0 & 3 & 0 \\ 0 & 6 & 2 & 1 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 4 \end{vmatrix} = 240.$$

Column development. What is developed in this paragraph also applies to the rows of a matrix. It is a matter of repeating and applying Proposition 3.7.

We propose to treat an arbitrary matrix $\mathbf{A} \in \mathscr{M}_3(\mathbb{K})$ and see if we can find Sarrus' rule. To do this, consider the matrix $\mathbf{A}$ defined by

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

And to do this, we propose to develop according to the first column (we could also choose to develop according to the first row). So we have :

$$
\begin{aligned}
det(\mathbf{A}) &= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \\
&= {\color{red}a_{11}}(-1)^{1+1}\begin{vmatrix} {\color{red}\cancel{a_{11}}} & \cancel{a_{12}} & \cancel{a_{13}} \\ \cancel{a_{21}} & a_{22} & a_{23} \\ \cancel{a_{31}} & a_{32} & a_{33} \end{vmatrix} + {\color{red}a_{21}}(-1)^{2+1}\begin{vmatrix} \cancel{a_{11}} & a_{12} & a_{13} \\ {\color{red}\cancel{a_{21}}} & \cancel{a_{22}} & \cancel{a_{23}} \\ \cancel{a_{31}} & a_{32} & a_{33} \end{vmatrix} + {\color{red}a_{31}}(-1)^{3+1}\begin{vmatrix} \cancel{a_{11}} & a_{12} & a_{13} \\ \cancel{a_{21}} & a_{22} & a_{13} \\ {\color{red}\cancel{a_{31}}} & \cancel{a_{32}} & \cancel{a_{33}} \end{vmatrix}, \\
&= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{21}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}, \\
&= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{21}(a_{12}a_{33} - a_{13}a_{32}) + a_{31}(a_{12}a_{23} - a_{13}a_{22}), \\
&= a_{11}a_{22}a_{33} + a_{12}a_{13}a_{31} + a_{21}a_{32}a_{13} - a_{11}a_{13}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}.
\end{aligned}
$$

Example 4.6. *We propose to calculate the determinant of the matrix A defined by*

$$
A = \begin{pmatrix} 3 & 1 & -2 & 4 \\ 0 & 0 & 5 & 1 \\ 0 & 2 & 0 & 0 \\ 3 & 6 & 0 & 1 \end{pmatrix}
$$

using transformations on the columns and expanding on the columns (or rows). This gives us

$$
\begin{aligned}
det(\mathbf{A}) &= \begin{vmatrix} 3 & 1 & -2 & 4 \\ 0 & 0 & 5 & 1 \\ 0 & 2 & 0 & 0 \\ 3 & 6 & 0 & 1 \end{vmatrix}, \\
&\downarrow \textit{ le déterminant reste inchangé par combinaison linéaires de lignes} \\
&= \begin{vmatrix} 3 & 1 & -2 & 4 \\ {\color{red}0} & 0 & 5 & 1 \\ {\color{red}0} & 2 & 0 & 0 \\ {\color{red}0} & 5 & 2 & -3 \end{vmatrix}_{L_4 \leftarrow L_4 - L_1}, \\
&\downarrow \textit{ we develop according to the first column} \\
&= 3\begin{vmatrix} 0 & 5 & 1 \\ 2 & 0 & 0 \\ 5 & 2 & -3 \end{vmatrix} \\
&\downarrow \textit{ we develop according to the second row} \\
&= 3 \times (-2)\begin{vmatrix} 5 & 1 \\ 2 & -3 \end{vmatrix}, \\
&\downarrow \textit{ we develop the determinant of size 2} \\
&= 3 \times (-2) \times (5 \times (-3) - 1 \times 2) = 102.
\end{aligned}
$$

4.3.1 Matrix inverse

Once we have calculated the determinant, we need to determine the expression of the comatrix, *i.e.* the cofactor matrix. To do this, we need to determine all the minors of our starting matrix. For an invertible matrix $\mathbf{A} \in \mathscr{M}_2(\mathbb{K})$ we have directly :

$$\text{si} \quad \mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad \text{alors} \quad A^{-1} = \frac{1}{det(A)} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11}. \end{pmatrix}.$$

For square matrices of size 2, the transpose of the cofactor matrix simply consists in swapping the place of the diagonal elements and changing the sign of the anti-diagonal elements.

Example 4.7. *Let us take our previous* $\mathbf{A}$ *matrix and determine its inverse. Since we have already calculated its determinant (*$det(\mathbf{A}) = 102$*), we simply need to determine the cofactor matrix* A_{ij}*. Once we've obtained this matrix, don't forget to transpose it to determine the inverse of matrix* $\mathbf{A}$*. We will simply calculate the values in the first row of the comatrix.*

- $A_{11} = (-1)^2 \Delta_{11} = \begin{vmatrix} 0 & 5 & 1 \\ 2 & 0 & 0 \\ 6 & 0 & 1 \end{vmatrix} = -2 \times \begin{vmatrix} 5 & 1 \\ 0 & 1 \end{vmatrix} = 10.$

  We will have to develop according to the second line to calculate the determinant of our 3 *square matrix.*
- $A_{12} = (-1)^3 \Delta_{12} = -\begin{vmatrix} 0 & 5 & 1 \\ 0 & 0 & 0 \\ 3 & 0 & 1 \end{vmatrix} = 0.$

  This is immediate, as our matrix contains a row with zeros only, so its determinant is zero.
- $A_{13} = (-1)^4 \Delta_{13} = \begin{vmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 3 & 6 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 2 \\ 3 & 6 \end{vmatrix} = -6.$

  We will have expanded according to the first line to obtain our new determinant of size 2*.*
- $A_{14} = (-1)^5 \Delta_{14} = -\begin{vmatrix} 0 & 0 & 5 \\ 0 & 2 & 0 \\ 3 & 6 & 0 \end{vmatrix} = -5 \times \begin{vmatrix} 0 & 2 \\ 3 & 6 \end{vmatrix} = 30.$

  We will have developed according to the first line again.

By calculating all the coefficients, we then show that the cofactor matrix is given by :

$$com(\mathbf{A}) = \begin{pmatrix} -10 & 0 & -6 & 30 \\ -4 & 0 & 18 & 12 \\ -127 & 51 & -15 & 75 \\ 44 & 0 & 6 & -30 \end{pmatrix}.$$

There is also a more practical way of calculating the inverse of a matrix, based on the fact that *the elementary operations on rows and columns are automorphisms, so they do not change the invertibility of a given matrix*. We can use this to find a sequence of elementary operations that will transform an invertible **A** matrix into the identity matrix. If we apply the same transformations to the identity matrix in parallel, we will be able to determine the inverse matrix of matrix **A**. We have two choices in terms of writing, but this choice will condition the work to be carried out and vice versa :

- if you want to perform elementary operations on the **rows of the A matrix**, work on the following *extended matrix*.

$$(\mathbf{A} \mid \mathbf{I}) = \left(\begin{array}{ccc|ccc} a_{11} & \cdots & a_{1n} & 1 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} & 0 & \cdots & 1 \end{array}\right),$$

  in order to obtain, via a succession of manipulations on the rows, an extended matrix of the form

$$(\mathbf{I} \mid \mathbf{A}') = \left(\begin{array}{ccc|ccc} 1 & \cdots & 0 & a'_{11} & \cdots & a'_{1n} \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & \cdots & 1 & a'_{n1} & \cdots & a'_{nn} \end{array}\right),$$

  where $\mathbf{A}' = \mathbf{A}^{-1}$ is the inverse of the matrix **A**.
- if we wish to perform elementary operations on **the columns of the matrix of A**, we will work on the following *extended matrix*.

$$\left(\frac{\mathbf{A}}{\mathbf{I}}\right) = \left(\begin{array}{ccc} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \\ \hline 1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 1 \end{array}\right).$$

Let us have a look on an example again.

Example 4.8. *Let* $\mathbf{A} \in \mathscr{M}_3(\mathbb{R})$ *be an invertible matrix (you can calculate its determinant to be sure), defined by :*

$$\mathbf{A} = \begin{pmatrix} 5 & 2 & 2 \\ 0 & 3 & 5 \\ 0 & 0 & 1 \end{pmatrix}.$$

The matrix **A** *is upper triangular in this example, and we propose to determine its inverse by performing elementary operations on the rows. Consider the following extended matrix :*

$$\left(\begin{array}{ccc|ccc} 5 & 2 & 2 & 1 & 0 & 0 \\ 0 & 3 & 5 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right).$$

We start by using the 1 *value at the bottom right of the matrix to be inverted. Don't forget to apply the same transformations to the identity matrix!*

$$\left(\begin{array}{ccc|ccc} 5 & 2 & 0 & 1 & 0 & -2 \\ 0 & 3 & 0 & 0 & 1 & -5 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} L_1 \leftarrow L_1 - 2L_3 \\ L_2 \leftarrow L_2 - 5L_3. \end{array}$$

Next, we will make the 1 *value appear in the second row and second column. Keeping in mind that the identity matrix must appear on the left, we iterate this same type of operation.*

$$\left(\begin{array}{ccc|ccc} 5 & 2 & 0 & 1 & 0 & -2 \\ 0 & 1 & 0 & 0 & 1/3 & -5/3 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) L_2 \leftarrow L_2/3,$$

$$\left(\begin{array}{ccc|ccc} 5 & 0 & 0 & 1 & -2/3 & 4/3 \\ 0 & 1 & 0 & 0 & 1/3 & -5/3 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) L_1 \leftarrow L_1 - 2L_2 \quad ,$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/5 & -2/15 & 4/15 \\ 0 & 1 & 0 & 0 & 1/3 & -5/3 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) L_1 \leftarrow L_1/5 \quad .$$

We thus have :

$$A^{-1} = \begin{pmatrix} 1/5 & -2/15 & 4/15 \\ 0 & 1/3 & -5/3 \\ 0 & 0 & 1 \end{pmatrix}.$$

Check your understanding

1. Are the following statements true or false ?
   (a) A column of a $\mathscr{M}_{n,p}(\mathbb{R})$ matrix is a $\mathbb{R}^p$. vector.
   (b) A row of a $\mathscr{M}_{n,p}(\mathbb{R})$ matrix is a $\mathbb{R}^n$. vector.
   (c) A non-zero matrix has no zero coefficient.
   (d) Every square matrix has an inverse.
   (e) Diagonal matrices are square matrices that are both lower triangular and upper triangular.
   (f) In $\mathscr{M}_n(\mathbb{R})$, matrix multiplication is commutative.
   (g) The product of a matrix and a null matrix is zero.
   (h) If $\mathbf{A}$ and $\mathbf{B}$ are matrices such that the product AB makes sense, then

$$\mathbf{AB} = \mathbf{0} \iff \mathbf{A} = \mathbf{0} \quad \text{or} \quad \mathbf{B} = \mathbf{0}.$$

   (i) If $\mathbf{A}$ is an invertible matrix of $\mathscr{M}_n(\mathbb{R})$ then for all $\mathbf{B}$

$$\mathbf{AB} = \mathbf{0} \iff \mathbf{B} = \mathbf{0}.$$

   (j) The rank of a $\mathscr{M}_{n,p}(\mathbb{R})$ matrix is less than $\inf(p, n)$.
   (k) The transpose of a product of matrices is the product of the transposes of these matrices.
   (l) The inverse of a product of invertible matrices is the product of the inverses of these matrices.

2. Compute the product $\mathbf{AB}$ where

$$\mathbf{A} = \begin{pmatrix} 1 & 3 & 0 \\ 0 & -2 & -1 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} 5 & -1 & -3 \\ 1 & 1 & 4 \\ -2 & 3 & 2 \end{pmatrix}.$$

3. Let $\mathbf{P} \in \mathscr{M}_n(\mathbb{R})$. Are the following propositions equivalent ? Does one imply the other ?
   (a) A : $\mathbf{P}$ is invertible, B : $\mathbf{P}$ is full rank.
   (b) A : $\mathbf{P}$ is invertible,
   B : For all $\mathbf{x} = (x_1, \ldots, x_n)^\top$, $\mathbf{Px} = \mathbf{0} \implies \mathbf{x} = \mathbf{0}$
   (c) A : $\mathbf{P}$ is invertible,
   B : no columns of $\mathbf{P}$ is equal to $\mathbf{0}$

(d) A : $\mathbf{P}$ is invertible,
B : The endomorphism of $\mathbb{R}^n$ canonically associated with P is an automorphism.

4. If $\mathbf{A}$ and $\mathbf{B}$ are two diagonal matrices, what is the value of the product $\mathbf{AB}$?

   $\mathbf{D}$ is a diagonal matrix, what is $\mathbf{D}^p$ for $p \in \mathbb{N}$?

5. Justify that two matrices $\mathbf{A}$ and $\mathbf{B}$ of $\mathscr{M}_{n,p}(\mathbb{R})$ are equal if and only if for all $\mathbf{x} \in \mathbb{R}^p$, we have $\mathbf{Ax} = \mathbf{Bx}$.

6. Let us consider $\mathbf{M} \in \mathscr{M}_n(\mathbb{K})$. Are the propositions A and B equivalent, and does one imply the other ?

   (a) A : $det(\mathbf{M}) = 0$,
   B : one column of $\mathbf{M}$ is equal to zero.
   (b) A : $det(\mathbf{M}) = 0$,
   B : two columns of $\mathbf{M}$ are proportional.
   (c) A : $det(\mathbf{M}) = 0$,
   B : one column of $\mathbf{M}$ is a linear combination of the other columns.

7. Compute the rank of the following matrices :

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 8 & 5 & 0 \\ -12 & 3 & 14 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 3 \\ 1 & 1 & 3 \end{pmatrix} \quad \text{and} \quad \mathbf{C} = \begin{pmatrix} 1 & 1 & 1-s \\ 1+s & -1 & 2 \\ 2 & -s & 3 \end{pmatrix}, \text{ where } s \in \mathbb{R}.$$

8. Show that the following matrix is invertible and determine its inverse

$$\begin{pmatrix} 0 & 1 & 2 \\ 2 & 2 & 3 \\ 4 & 0 & 1 \end{pmatrix}.$$

9. Let $(a, b) \in^2$. Calculate the determinant of the following matrices and specify whether they are invertible and on what condition(s).

$$\mathbf{A} = \begin{pmatrix} 1 & a \\ 1 & b \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 1 & a & b \\ a & 1 & b \\ b & a & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{C} = \begin{pmatrix} a & 0 & 0 & b \\ b & 1 & 1 & a \\ a & 1 & 1 & b \\ 0 & 2 & 1 & a \end{pmatrix}.$$