Optimization & Operational Reasearch
Mock Exam

Exercise 1 : True of False (No justification)

Answer *True* or *False* to the following assertions.

1. Consider the function $f : \mathbb{R}^n \to \mathbb{R}$ defined by :

$$f(x) = \frac{1}{2} x^T A x - b^T x,$$

where $A \in \mathbb{R}^{n \times n},\ b \in \mathbb{R}^n$.

(a) f is quadratic function.

(b) For any matrices $A \in \mathbb{R}^{n \times n}$ f is convex.

(c) The function g defined by $g(x) = \exp(f(x))$ is convex.

(d) The function h defined by $h(x) = \exp(-f(x))$ is convex.

2. We consider a function f for which it exists a vector u such that $\nabla f(u) = 0$.

(a) u is an extremum (minimum or maximum) of the function f.

(b) u is a global minimum of f if its last is convex.

3. We consider a function $f : \mathbb{R}^2 \to \mathbb{R}$ and we denote by H its second order derivative and λ_1 and λ_2 the eigenvalues of H.

(a) f is concave if λ_1 and λ_2 are negative.

(b) f is concave if $Trace(H) < 0$ and $det(H) < 0$.

(c) f is convex if both Trace and Determinant are positive.

(d) What about the last assertion if $f : \mathbb{R}^n \to \mathbb{R}$ where $n > 2$.

4. Let us consider ton convex sets C_1 and C_2 and any straight line D.

(a) The intersection $C = C_1 \cap C_2$ is always convex.

(b) The union $C = C_1 \cup C_2$ is always convex.

5. About the algorithms

(a) The Gradient Descent with a fix step always converges.

(b) Finding the Optimal step consists of solving an other minimization problem at each iteration of Gradient Descent algorithm.

(c) The Newton's method can be applied even if the function is not convex.

Exercise 2 : Convexity and Convex Set

1. We say that a function $f : \mathbb{R} \to \mathbb{R}_+^*$ is log-convex if the function $g : \mathbb{R} \to \mathbb{R}$ defined by : $g(x) = \log(f(x))$ is convex.
   We suppose that f is **twice differentiable** and log-convex. Show that f is convex.
2. We consider the set C defined by

$$C = \{x \in \mathbb{R} \mid x^2 - 6x + 5 \leq 0.\}$$

   Show that this set is convex.
3. Show that the hyperbolic set $\{x \in \mathbb{R}_+^n \mid \prod_{i=1}^n x_i \geq 1\}$ is convex.
   Hint : you can first show that, for all $x, y \in \mathbb{R}_{++}$ and $\theta \in [0, 1]$ we have : $x^\theta y^{1-\theta} \leq \theta x + (1 - \theta)y$.

Exercise 3 : Optimization and Algorithm

We consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by :

$$f(x, y) = 5x^2 + 3y^2 - 6xy + 2x + 3y + 1$$

1. Is the function f convex ? Why ?
2. Find the solution(s) of Euler's Equation : $\nabla f(x, y) = (0, 0)$.
3. What is the global minimum of the function ?
4. We set $u_0 = (x, y)^{(0)} = (1, 1)$, the initial point of the Gradient Descent Algorithm with a learning rate $\rho = 0.5$.
   (a) First recall what this method consists of.
   (b) Calculate u_1 and u_2. What do you think of the choice of the learning rate ?
5. Can we apply the Newton's Method using the same initial point ? Why ?
6. Would you apply the Gradient Descent with Optimal step in this case and why ? Recall how we can compute the optimal learning rate if we are at the point $u_k = (x, y)^{(k)}$ (give the value of the matrix A and the vector b as they were defined in class).